DIRICHLET INTEGRALS AND H_4 NORMS OF ANALYTIC FUNCTIONS

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For every analytic function f in the unit disc U with $f(0)=0$, the inequality $\|f\|_4 \leqq \left(\frac{1}{\pi}\int\int_U |f'(z)|^2 dxdy\right)^{1/2}$ is proved. As a corollary it is shown that if $h_4(z)$ denotes the least harmonic majorant of $|z|^4$ in a simply-connected domain D in the complex plane with $0 \in D$, then the inequality $h_4(0) \leqq \left\{\frac{1}{\pi}\text{area}(D)\right\}^{1/2}$ holds. This gives a partially affirmative answer to a conjecture presented by M. Sakai.

Key words: Dirichlet integral/H_p norm/least harmonic majorant/inner function

1. Introduction

Let $U=\{|z|<1\}$ be the unit disc in the complex plane. Let p be a positive number. For a function f analytic in U, the H_p *norm* $\|f\|_p$ of f is defined by

$$\|f\|_p = \left\{\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right\}^{1/p}. \qquad (1.1)$$

Let $H_{f,p}(z)$ denote the *least harmonic majorant* of $|f(z)|^p$ in U. It is well-known (see, for example, the page 28 of the standard textbook on H_p spaces by P. L. Duren[5]) that the following equality holds for any f:

$$\|f\|_p = \{H_{f,p}(0)\}^{1/p}. \qquad (1.2)$$

The space of all functions f analytic in U for which $\|f\|_p$ are finite is denoted by $H_p(U)$ and called Hardy classes. An analytic function φ in U with $|\varphi(z)| \leqq 1$ for $z \in U$ is called an *inner function* if $\|\varphi\|_p = 1$ (for some p or equivalently for any p). The *Dirichlet integral* $D(f)$ of f is defined by

$$D(f) = \frac{1}{\pi}\int\int_U |f'(z)|^2 dxdy,$$

where $z=x+iy$. By $A(f)$ we denote the $1/\pi$ times of the image area of f, that is,

$$A(f) = \frac{1}{\pi}\text{area}\{f(U)\}. \qquad (1.3)$$

Obviously, $A(f) \leqq D(f)$, and when $D(f) < +\infty$ equality occurs in the inequality if and only if f is univalent in U.

From now on and throughout the present note, let f denote an analytic function in U with $f(0)=0$. In 1972, Alexander, Taylor and Ullman[3] proved the following inequality holds for such f:

$$\|f\|_2 \leqq A(f)^{1/2}. \qquad (1.4)$$

Since then, various proofs of the inequality have been given by many authors[1)2)4)7)], and it is known that equality occurs in (1.4) if and only if f is a constant multiple of an inner function. Here and throughout the present note, "equality occurs" means "the both sides are finite and equality occurs."

Recently Sakai[10] considered related problems of more general setting including higher dimensional cases, and as a corollary to his main theorem, he improved (1.4) to

$$\|f\|_p \leqq A(f)^{1/2} \qquad (1.5)$$

for $0 < p \leqq 2+1/2$, and he showed that when $0 < p < 2+1/2$ equality occurs in (1.5) if and only if f is a constant multiple of an inner function. He also conjectured that the inequality (1.5) would hold for $0 < p \leqq 4$.

In the present note, we show that the inequality

$$\|f\|_p \leqq D(f)^{1/2} \qquad (1.6)$$

holds for any p with $0 < p \leqq 4$, where equality occurs if and only if $f(z)=cz$ with a constant c. For the proof we use the expressions of $\|f\|_2$ and

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$D(f)$ by the *Taylor coefficients of* f and an elementary inequality on sums of finite real numbers. As a corollary, we see that when the image $f(U)$ is simply-connected Sakai's conjecture mentioned above is valid.

2. Dirichlet integrals

It is well-known that f is expressed as a power series about 0 in U:

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{2.1}$$

Simple calculations show (see, for example, the pages 8 and 108 of the Duren's book[5] cited above) that the H_2 norm $\|f\|_2$ and the Dirichlet integral $D(f)$, respectively, of f are expressed by the coefficients $\{a_n\}$:

Lemma 1. $\{\|f\|_2\}^2 = \sum_{n=1}^{\infty} |a_n|^2.$

Lemma 2. $D(f) = \sum_{n=1}^{\infty} n|a_n|^2.$

The following is an elementary inequality, which is the restatement of *Jensen's inequality* (see, for example, the page 62 of Rudin's book[8] on real and complex analysis) for the convex function $\lambda(t) = t^2$, or which is a simple consequence of the *Schwarz inequality*:

Lemma 3. *For any N real numbers* $x_1, x_2, \cdots, x_N$, *the inequality*

$$\left(\sum_{j=1}^{N} x_j\right)^2 \leqq N \sum_{j=1}^{N} x_j^2$$

holds, where equality occurs if and only if $x_1 = x_2 = \cdots = x_N$.

Now we state our theorem and prove it:

Theorem. *If* f *is an analytic function in* U *with* $f(0) = 0$, *then the inequality*

$$\|f\|_p \leqq D(f)^{1/2} \tag{2.2}$$

holds for $0 < p \leqq 4$, *and equality occurs in the inequality if and only* if $f(z) = cz$ *with a constant* c.

Proof. Since $\|f\|_p$ is nondecreasing with p, it is sufficient to prove the inequality for $p = 4$, that is

$$\|f\|_4 \leqq D(f)^{1/2}. \tag{2.3}$$

Set $g(z) = \{f(z)\}^2$, then we have from (2.1)

$$g(z) = \sum_{n=2}^{\infty} b_n z^n,$$

with

$$b_n = \sum_{s+t=n} a_s a_t \equiv a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1. \tag{2.4}$$

On noting the definition (1.1) of H_p norms and applying Lemma 1 to g, we see

$$\{\|f\|_4\}^4 = \{\|g\|_2\}^2 = \sum_{n=2}^{\infty} |b_n|^2. \tag{2.5}$$

Applying Lemma 3 with $x_j = |a_j a_{n-j}|$ for $j = 1, 2, \ldots, n-1$ and $N = n-1$, we obtain from (2.4)

$$|b_n|^2 \leqq \left(\sum_{s+t=n} |a_s a_t|\right)^2, \leqq (n-1) \sum_{s+t=n} |a_s|^2 |a_t|^2, \tag{2.6}$$

for every integer $n \geqq 2$. For any positive integers s and t, we see $st \geqq s+t-1$, since $st-s-t+1 = (s-1)(t-1) \geqq 0$, and hence $n-1 \leqq st$ if $s+t = n$. Therefore we see by (2.6)

$$|b_n|^2 \leqq \sum_{s+t=n} st|a_s|^2|a_t|^2. \tag{2.7}$$

By combining (2.5) and (2.7) and using Lemma 2, we obtain

$$\{\|f\|_4\}^4 \leqq \sum_{n=2}^{\infty} \sum_{s+t=n} st|a_s|^2|a_t|^2 = \left(\sum_{n=1}^{\infty} n|a_n|^2\right)^2 = D(f)^2, \tag{2.8}$$

which is nothing but the asserted inequality (2.2).

Next we prove the equality statement. Suppose that $a_k \neq 0$ for some $k > 1$. Since $k^2 > 2k-1$, we easily see that for $n = 2k$ the inequality (2.7) becomes a strict one, and hence so is (2.8). Therefore, we see that if equality occurs in (2.2), then $a_k = 0$ for $k = 2, 3, \ldots$, which means $f(z) = cz$ with a constant c.

Conversely, if $f(z) = cz$, the simple calculations show $\|f\|_p = |c|$ and $D(f) = |c|^2$, and hence equality holds in (2.2). This completes the proof of the theorem.

3. The least harmonic majorant of $|z|^p$

Let D be a domain in the complex plane with $0 \in D$. By $h_p(z)$ we denote the least harmonic majorant of $|z|^p$ in D, where we set $h_p(z) \equiv +\infty$ if $|z|^p$ admits no harmonic majorants in D.

The author[7] proved the inequality

$$h_2(0) \leqq \frac{1}{\pi}\text{area}(D)$$

in order to give a proof of the Alexander-Taylor-Ullman inequality (1.4). Sakai[10] improved this to

$$h_p(0) \leqq \left\{\frac{1}{\pi}\text{area}(D)\right\}^{p/2}$$

for $0 < p \leqq 2+1/2$.

As a corollary to our theorem, we obtain

Corollary 1. *If D is simply-connected, then the inequality*

$$h_p(0) \leqq \left\{\frac{1}{\pi}\text{area}(D)\right\}^{p/2} \tag{3.1}$$

holds for $0 < p \leqq 4$. Equality occurs in (3.1) *if and only if D is a disc centered at* 0.

Proof. We may assume that D is conformally equivalent to U, since if otherwise area$(D) = +\infty$. Let g be a conformal map of U onto D with $g(0) = 0$. Then we easily see that $h_p(g(z))$ coincides with the least harmonic majorant $H_{g,p}(z)$ of $|g(z)|^p$ in U, and hence we obtain by noting (1.2)

$$\|g\|_p = h_p(0)^{1/p}. \tag{3.2}$$

The univalency of g implies

$$D(g) = A(g) = \frac{1}{\pi}\text{area}(D). \tag{3.3}$$

Applying the theorem to g, we have

$$\|g\|_p \leqq D(g)^{1/2} \tag{3.4}$$

for $0 < p \leqq 4$. Combining (3.2), (3.3) and (3.4), we obtain the asserted inequality (3.1).

From the equality statement of the theorem, we see that equality occurs in (3.4), or equivalently in (3.1), if and only if $g(z) = cz$ with a constant c. This is equivalent to the fact that D is a disc centered at 0.

Corollary 2. *If f is an analytic function in U with $f(0) = 0$ such that $f(U) = D - E$, where D is a simply-connected domain and E is a closed set of area zero, then the inequality*

$$\|f\|_p \leqq A(f)^{1/2} \tag{3.5}$$

holds for $0 < p \leqq 4$. Equality occurs in (3.5) *if and only if $f(z) = c\varphi(z)$ with a constant c and an inner function φ.*

Proof. Let g be a conformal map of U onto D with $g(0) = 0$, and set $\varphi(z) = g^{-1}(f(z))$. Then we see $f(z) = g(\varphi(z))$, which means f is subordinate to g. By *Littlewood's subordination principle* (see, for example, the page 10 of the Duren's book[5]), we see

$$\|f\|_p \leqq \|g\|_p. \tag{3.6}$$

Applying the theorem to g, we see

$$\|g\|_p \leqq D(g)^{1/2} = A(g)^{1/2} = A(f)^{1/2} \tag{3.7}$$

for $0 < p \leqq 4$, since g is univalent and we assumed that E is of area zero. Combining (3.6) and (3.7), we obtain the asserted inequality (3.5).

Suppose that equality occurs in (3.5). Then, equality occurs in both (3.6) and (3.7). By Ryff's theorem[9] on subordination for H_p functions, the equality in (3.6) implies that φ is an inner function. From the equality statement of our theorem, we see that the equality in (3.7) implies $g(z) = cz$ and hence $f(z) = c\varphi(z)$, with a constant c and an inner function φ.

Conversely, if f is of such a form, we see

$$\|f\|_p = A(f)^{1/2} = |c|,$$

since any inner function assumes every point in U possibly except for a set of *capacity* zero by a *theorem of Frostman* (see, for example, the page 80 of the standard textbook[6] on bounded analytic functions by J.B. Garnett), and a set of capacity zero is evidently of area zero.

4. Concluding remarks

The inequality (1.6) does not holds for any $p > 4$. In fact, consider $f(z) = z + cz^2$ with a constant c, then we easily see by Lemma 2

$$D(f) = 1 + 2|c|^2. \tag{4.1}$$

Next, in order to estimate $\|f\|_p$, let $g(z) = f(z)/z \equiv 1 + cz$, which has no zeros in U for $|c| < 1$. Therefore, the branch $\psi(z)$ of $\{g(z)\}^{p/2}$ with $\psi(0)=1$ is well-defined in U and

$$\psi(z)=1+\frac{p}{2}cz+\cdots.$$

On noting (1.1) and Lemma 1, we see

$$\begin{aligned}\|f\|_p &= \|g\|_p=\{\|\psi\|_2\}^{2/p}\\ &=\left(1+\frac{p^2}{4}|c|^2+\cdots\right)^{1/p}\\ &=1+\frac{p}{4}|c|^2+\cdots. \qquad (4.2)\end{aligned}$$

By comparing (4.1) with (4.2), we obtain

$$\|f\|_p > D(f)^{1/2},$$

if $|c|$ is sufficiently small.

It is plausible that the inequality (3.5) or equivalently (3.1) also holds generally for $0<p\leqq 4$, but we do not know as yet whether this is true or not.

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